Gauge fields and Kaluza-Klein theory

Correspondance potential A_μ - connection and field $F_{\mu\nu}$ - curvature

In a loop (+dx,+dy,-dx,-dy) the phase increases by :
$A_x\, dx + (A_y + \partial_x A_y\, dx)\, dy - (A_x + \partial_y A_x\, dy)\, dx - A_y\, dy$
$= (\partial_x A_y - \partial_y A_x)\, dx\, dy$
$= F_{xy}\, dx\, dy$

Kaluza-Klein theory

The vielbeins $e^a{}_\mu$ are the matrices of base changing between curved and locally flat coordinates.
The metric tensor $g_{\mu\nu}$ in curved coordinates is :
$g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu\, e^b{}_\nu$
where η_{ab} is the metric tensor of the flat space-time (1 -1 -1 -1 on the diagonal, 0 elsewhere).

According to Kaluza-Klein theory, gauge fields are equivalent to gravitation (curvature of space-time according to general relativity) in compactified dimensions.

The vielbein is :

$$e^a{}_\mu = \begin{pmatrix} e'^{a'}{}_\mu & 0 \\ e''^{a''}{}_{\nu''}\, A^{\nu''}{}_\mu & e''^{a''}{}_{\mu''} \end{pmatrix}$$

where $e'^{a'}{}_\mu$ is the vielbein of the space-time, $e''^{a''}{}_{\mu''}$ is the vielbein of the compactified dimensions, and $A^{\nu''}{}_\mu$ is the potential.

We can number the dimensions of space-time from 0 to 3 (0=t, 1=x, 2=y, 3=z) and the compactified dimensions from 4 to d-1, and consider all the tensor defined for indices varying from 0 to d-1, completed with zeros, for example :

$$A^m{}_\mu = \begin{pmatrix} 0 & 0 \\ A^m{}_\mu & 0 \end{pmatrix}$$

with indices $\mu = 0..3$ and $m = 4..d-1$; $A^m{}_\mu$ is 0 if μ is not in 0..3 or m is not in 4..d-1.

With this convention, we have :
$e^a{}_\mu = e'^a{}_\mu + e''^a{}_n\, A^n{}_\mu + e''^a{}_\mu$
$= e'^{a'}{}_\mu + e''^a{}_n\, (A^n{}_\mu + \delta^n{}_\mu)$
$= e'^a{}_\mu + e''^a{}_\mu + e''^a{}_m\, A^m{}_\mu$

Then the metric tensor in curved coordinates is :
$g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu\, e^b{}_\nu$
$= \eta_{ab}\, (e'^a{}_\mu + e''^a{}_\mu + e''^a{}_m\, A^m{}_\mu)\, (e'^b{}_\nu + e''^b{}_\nu + e''^b{}_n\, A^n{}_\nu)$
$= \eta_{ab}\, e'^a{}_\mu\, e'^b{}_\nu + \eta_{ab}\, e'^a{}_\mu\, e''^b{}_\nu + \eta_{ab}\, e'^a{}_\mu\, e''^b{}_n\, A^n{}_\nu + \eta_{ab}\, e''^a{}_\mu\, e'^b{}_\nu + \eta_{ab}\, e''^a{}_\mu\, e''^b{}_\nu + \eta_{ab}\, e''^a{}_\mu\, e''^b{}_n\, A^n{}_\nu + \eta_{ab}\, e''^a{}_m\, A^m{}_\mu\, e'^b{}_\nu + \eta_{ab}\, e''^a{}_m\, A^m{}_\mu\, e''^b{}_\nu + \eta_{ab}\, e''^a{}_m\, A^m{}_\mu\, e''^b{}_n\, A^n{}_\nu$
Since $\eta_{ab}\, e'^a{}_\mu\, e''^b{}_\nu = 0$ because η_{ab} is not 0 only if a = b, $e'^a{}_\mu$ is not zero only if a is in 0..3 and $e''^b{}_\nu$ is not zero only if b is in 4..d-1, and same for $\eta_{ab}\, e''^a{}_\mu\, e'^b{}_\nu$ we have :
$g_{\mu\nu} = \eta_{ab}\, e'^a{}_\mu\, e'^b{}_\nu + \eta_{ab}\, e''^a{}_\mu\, e''^b{}_\nu + \eta_{ab}\, e''^a{}_\mu\, e''^b{}_n\, A^n{}_\nu + \eta_{ab}\, e''^a{}_m\, A^m{}_\mu\, e''^b{}_\nu + \eta_{ab}\, e''^a{}_m\, A^m{}_\mu\, e''^b{}_n\, A^n{}_\nu$
$= g'_{\mu\nu} + g''_{\mu\nu} + g''_{\mu n}\, A^n{}_\nu + g''_{m\nu}\, A^m{}_\nu + g''_{mn}\, A^m{}_\mu\, A^n{}_\nu$