Fruitful and helpful ordinal functions

Harold Simmons

School of Mathematics

The University, Manchester, England

hsimmons @ manchester.ac.uk

Abstract

In [7] I described a method of producing ordinal notations 'from below' (for countable ordinals up to the Howard ordinal) and compared that method with the current popular 'from above' method which uses a collapsing function from uncountable ordinals. This 'from below' method employs a slight generalization of the normal function – the fruitful functions – and what seems to be a new class of functions – the helpful functions – which exist at all levels of the function space hierarchy over ordinals. Unfortunately I was rather sparing in my description of these classes of functions. In this paper I am much more generous. I describe the properties of the helpful functions on all finite levels and, in the final section, indicate how they can be used to simplify the generation of ordinal notations.

The main aim of this paper is to fill in the details missing from [7]. The secondary aim is to indicate what can be done with helpful functions. Fuller details of this development will appear elsewhere.

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1 Preamble

In [7] I compared two methods of generating notations for countable ordinals: the standard method using a collapsing function; and a less standard but historically older method.

In the review [5] of [7] the reviewer quite rightly criticized me for omitting certain proofs from [7]. The paper doesn't even contain the definition of a crucial central notion. This paper corrects that omission, and with [7] forms a self contained account.

Let me outline what this paper and [7] are about.

Let Ω be the least uncountable ordinal. We wish to name as many ordinals $\alpha < \Omega$ as possible. Let

$$\Omega^+ = \Omega^{\Omega^{\Omega^{\cdot^{\cdot^{\cdot}}}}} = \epsilon_{\Omega+1}$$

be the next critical ordinal beyond Ω. An ordinal ν is critical if $\nu = \omega^\nu$.

In [7] I described the standard method of generating notations as 'from above'. By that I merely meant that larger ordinals, up to Ω^+, are used to index the generation of smaller ordinals, well below Ω. The collapsing function

$$\psi : [0, \Omega^+) \longrightarrow [0, \Omega)$$

sends larger ordinals to smaller ordinals.

I described the alternative method of generating notations as 'from below'. By that I merely meant that ordinals already generated are used to index the next phase of generation. As each phase peters out some new gadget has to be conjured up to keep the process going. It was Veblen in [9] who first used this method to generate ordinal notations, but the idea goes right back to Archimedes in [1] who used this method to generate notations for natural numbers.

The new gadgets I employed are certain fixed point extractors, and to produce these I used what I called helpful functions. It is an account of helpful functions that is missing from [7]. This paper fills that gap.

The main results of this paper are contained in Sections 3, 4, and 5. Let me give a brief description of that material.

Let $\mathbb{O}\mathrm{rd}$ be the set of countable ordinals. Thus $\mathbb{O}\mathrm{rd} = [0, \Omega)$. From now on in this paper 'ordinal' means 'countable ordinal', a member of $\mathbb{O}\mathrm{rd}$. Let

$$\mathbb{O}\mathrm{rd}' = (\mathbb{O}\mathrm{rd} \longrightarrow \mathbb{O}\mathrm{rd})$$

be the set of all ordinal functions. The notion of a normal function $f : \mathbb{O}\mathrm{rd}'$ is a central component of any method of generating ordinal notations. Essentially we have to harvest the fixed points of such a function f. For what we do here it turns out that the class of normal functions is slightly too small. Thus we use the slightly larger class $\mathbb{F}\mathrm{ruit} \subseteq \mathbb{O}\mathrm{rd}'$ of fruitful functions. These functions $f \in \mathbb{F}\mathrm{ruit}$ are fruitful because they provide many fixed points, all of which are critical. The class $\mathbb{F}\mathrm{ruit}$ is defined and discussed in the first part of Section 3.

In general, one fruitful (or normal) function is not enough. To generate a decent stretch of ordinals we need several fruitful functions. The class $\mathbb{H}\mathrm{elp} \subseteq \mathbb{O}\mathrm{rd}'$ of helpful functions makes it easier to grow the required fruitful functions. Each helpful function $h \in \mathbb{H}\mathrm{elp}$ can be iterated

$$\alpha \longmapsto h^\alpha \zeta$$

through $\mathbb{O}\mathrm{rd}$ (for any given input ζ) and this is a fruitful function. This format makes the arithmetic of the fruitful functions easier to handle. The class $\mathbb{H}\mathrm{elp}$ is defined and discussed in the latter part of Section 3.

This method of generating ordinals 'from below' depends on producing enough members of $\mathbb{H}\mathrm{elp}$. To do that we use certain higher order functions. These are also called helpful because the defining properties of the whole family have a certain uniformity. This larger family is described in Section 4, and a particular family of such functions is described in Section 5.

Section 2 gathers together all the bits and pieces that we need. Some of this is a repeat of Section 2 of [7].

Finally, in Section 6, I give a brief historical account of the various phases in the development of ordinal notations. I indicate how each phase can be seen in terms of helpful functions, each phase using such functions at higher and higher levels. This final section is merely an indication of what can be done with helpful functions. A full development will appear elsewhere.

To conclude this preamble I indicate where the proofs missing from [7] can be found in this paper.

Result from [7]		is proved here as:
2.6	(a)	2.4 and 4.4
2.6	(b)	4.1(Help1)
2.6	(c)	3.13(b) and 4.6(b)
2.7	(1)	3.13(c)
2.7	(> 1)	4.6(c)
2.9		3.10(b) and 4.6
2.13		5.10

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I thank the referee for his very careful reading of several versions of this paper. He made several important observations and spotted a gap in the proof of a crucial result.

2 Background material

Let $\mathbb{O}\mathrm{rd}$ be the set of countable ordinals. Except for a brief mention towards the end of the paper, all the ordinals we meet belong to $\mathbb{O}\mathrm{rd}$. Thus it is safe to let 'ordinal' mean 'countable ordinal'. We use various ordinal functions

$$f : \mathbb{O}\mathrm{rd} \longrightarrow \mathbb{O}\mathrm{rd}$$

as well as higher level versions of such functions. To handle these we set up a bit of notation.

For an arbitrary set $\mathbb{S}$ let

$$\mathbb{S}' = (\mathbb{S} \longrightarrow \mathbb{S})$$

the set of functions on $\mathbb{S}$. This construction $(\cdot)'$ can be iterated.

2.1 DEFINITION. The chain $\mathbb{O}\mathrm{rd}^{(\cdot)}$ of spaces is generated by

$$\mathbb{O}\mathrm{rd}^{(0)} = \mathbb{O}\mathrm{rd} \qquad \mathbb{O}\mathrm{rd}^{(r+1)} = \mathbb{O}\mathrm{rd}^{(r)\prime}$$

for each $r < \omega$. ■

Thus $\mathbb{O}\mathrm{rd}^{(0)}$ is just the space $\mathbb{O}\mathrm{rd}$ of ordinals, and $\mathbb{O}\mathrm{rd}^{(1)}$ is the space $\mathbb{O}\mathrm{rd}'$ of ordinal functions, and so on. It seems that members of $\mathbb{O}\mathrm{rd}^{(l+2)}$ are rarely used, but we will meet several in this paper. Notice that this space $\mathbb{O}\mathrm{rd}^{(l+2)}$ can be decomposed as

$$\mathbb{O}\mathrm{rd}^{(l+2)} = \mathbb{O}\mathrm{rd}^{(l+1)} \rightarrow \mathbb{O}\mathrm{rd}^{(l)} \rightarrow \cdots \rightarrow \mathbb{O}\mathrm{rd}' \rightarrow \mathbb{O}\mathrm{rd} \rightarrow \mathbb{O}\mathrm{rd}$$

(where punctuating brackets should be inserted in the obvious way). In particular, each function $G : \mathbb{O}\mathrm{rd}^{(l+2)}$ must receive successive inputs

$$g : \mathbb{O}\mathrm{rd}^{(l+1)},\ g_l : \mathbb{O}\mathrm{rd}^{(l)},\ \ldots,\ g_1 : \mathbb{O}\mathrm{rd}',\ \zeta : \mathbb{O}\mathrm{rd}$$

to produce

$$Gg : \mathbb{O}\mathrm{rd}^{(l+1)},\ Ggg_l : \mathbb{O}\mathrm{rd}^{(l)},\ \dots,\ Ggg_l\cdots g_1 : \mathbb{O}\mathrm{rd}'$$

and then return its eventual output $Ggg_l\cdots g_1\zeta \in \mathbb{O}\mathrm{rd}$.

The space $\mathbb{O}\mathrm{rd}$ is linearly ordered and carries an actual supremum operation $\bigvee$ which converts each countable subset $X \subseteq \mathbb{O}\mathrm{rd}$ into its least upper bound $\bigvee X$. There is a formal way to lift this operation to higher levels.

The following definition could be given in a more compact form, but it is safer to format it as a kind of recursion.

2.2 DEFINITION. (base) For each non-empty, countable subset $\mathcal{G} \subseteq \mathbb{O}\mathrm{rd}'$ the function $\bigvee\mathcal{G} : \mathbb{O}\mathrm{rd}'$ is given by

$$\Big(\bigvee\mathcal{G}\Big)\zeta = \bigvee\{g\zeta \mid g \in \mathcal{G}\}$$

(for $\zeta \in \mathbb{O}\mathrm{rd}$). We call this function $\bigvee\mathcal{G}$ the pointwise supremum of $\mathcal{G}$.

(raise) For each $l < \omega$ and each non-empty, countable subset $\mathcal{G} \subseteq \mathbb{O}\mathrm{rd}^{(l+2)}$ the function $\bigvee\mathcal{G} : \mathbb{O}\mathrm{rd}^{(l+2)}$ is given by

$$\Big(\bigvee\mathcal{G}\Big)g = \bigvee\{Gg \mid G \in \mathcal{G}\}$$

(for $g \in \mathbb{O}\mathrm{rd}^{(l+1)}$). We call this function $\bigvee\mathcal{G}$ the pointwise supremum of $\mathcal{G}$. ■

To explain what is going on let us temporarily write $\bigvee^{(l)}$ for the gadget constructed on $\mathbb{O}\mathrm{rd}^{(l)}$. Thus $\bigvee^{(0)}$ is the actual supremum operation on $\mathbb{O}\mathrm{rd}$, and then we generate $\bigvee^{(1)}, \bigvee^{(2)}, \dots, \bigvee^{(l)}, \dots$ in turn by recursion on l. Later we work with a subclass $\mathbb{H}^{(l)} \subseteq \mathbb{O}\mathrm{rd}^{(l)}$ which can be partially ordered with $\bigvee^{(l)}$ as a supremum operation.

For $\mathcal{G} \subseteq \mathbb{O}\mathrm{rd}^{(l+2)}$ the construction of $\bigvee^{(l+2)}\mathcal{G}$ can be unravelled as

$$\Big(\bigvee^{(l+2)}\mathcal{G}\Big)gg_l\cdots g_1\zeta = \bigvee\{Ggg_l\cdots g_1\zeta \mid G \in \mathcal{G}\}$$

using the actual supremum operation on $\mathbb{O}\mathrm{rd}$.

Ordinal iterations of function $g \in \mathbb{O}\mathrm{rd}'$ are standard fare. The pointwise supremum enables us to lift this to higher levels.

2.3 DEFINITION. (a) For each $l < \omega$ and each $g : \mathbb{O}\mathrm{rd}^{(l+1)}$, the ordinal iterates $g^\bullet$ of g are generated by

$$g^0 = id \qquad g^{\alpha+1} = g \circ g^\alpha \qquad g^\lambda = \bigvee\{g^\alpha \mid \alpha < \lambda\}$$

for each $\alpha \in \mathbb{O}\mathrm{rd}$ and limit ordinal $\lambda \in \mathbb{O}\mathrm{rd}$. (Here id is the identity function on $\mathbb{O}\mathrm{rd}^{(l)}$.)

(b) For each $l < \omega$ a class $\mathbb{S} \subseteq \mathbb{O}\mathrm{rd}^{(l+1)}$ is smooth if $f \circ g \in \mathbb{S}$ for each $f, g \in \mathbb{S}$, and $\bigvee\mathcal{G} \in \mathbb{S}$ for each non-empty and countable $\mathcal{G} \subseteq \mathbb{S}$. ■

Again, for the moment, treat this as nothing more than a definition. In general the ordinal iterates of g (as defined here) may not behave as you think they should. The notion of a smooth class is a way of calming down some of the wilder behaviour.

2.4 LEMMA. *Suppose $\mathbb{S} \subseteq \mathbb{O}\mathrm{rd}^{(l+1)}$ is smooth and $g \in \mathbb{S}$. Then $g^\alpha \in \mathbb{S}$ for each non-zero ordinal α.*

We will construct several smooth classes, most at higher levels. However, for the first examples we stick with $\mathbb{O}\mathrm{rd}'$.

3 Low level functions

In this section we look at standard ordinal functions of type $\mathbb{O}\mathrm{rd}'$. We isolate two classes of such functions. The class of fruitful functions $\mathbb{F}\mathrm{ruit} \subseteq \mathbb{O}\mathrm{rd}'$ forms a rather mild expansion of the usual class of a normal function. This class $\mathbb{F}\mathrm{ruit}$ is more amenable and, in particular, it is smooth. The class $\mathbb{H}\mathrm{elp} \subseteq \mathbb{O}\mathrm{rd}'$ of helpful functions on this level is also smooth. It is the interaction between $\mathbb{F}\mathrm{ruit}$ and $\mathbb{H}\mathrm{elp}$ that interests us here.

Thus there are three main classes of ordinal functions that we meet: general functions, fruitful functions, and helpful functions. Usually we write

f for a fruitful function $\quad$ g for a general function $\quad$ h for a helpful function

to indicate which kind of function is being used. This is a convenient informal convention which, of course, may be broken at times. Fruitful functions are so called because they have lots of fixed points each of which is critical. The helpful functions enable us to produce fruitful functions, and hence generate critical ordinals.

(Actually, this convention came about because for a long time I couldn't remember the difference between fruitful and helpful – which I was then calling something else. I had to invent this little trick to keep my sanity. I'm not sure it worked.)

To begin the analysis we first isolate a smooth class $\mathbb{IM} \subseteq \mathbb{O}\mathrm{rd}$ which includes both $\mathbb{F}\mathrm{ruit}$ and $\mathbb{H}\mathrm{elp}$.

We are interested in various combinations of standard property of functions $g : \mathbb{O}\mathrm{rd}'$. Most of these have names.

3.1 DEFINITION. A function $g : \mathbb{O}\mathrm{rd}'$ is, respectively

(i)	inflationary	if	$\alpha \leq g\alpha$
(si)	strictly inflationary	if	$\alpha < g\alpha$
(m)	monotone	if	$\alpha \leq \beta \Rightarrow g\alpha \leq g\beta$
(sm)	strictly monotone	if	$\alpha < \beta \Rightarrow g\alpha < g\beta$
(b)	big	if	$\omega^\alpha \leq g\alpha$ (except possibly for $\alpha = 0$)
(sb)	strictly big	if	$g\alpha$ is critical
(c)	continuous	if	$g(\bigvee A) = \bigvee g[A]$

for all ordinals α, β, and each non-empty countable set A of ordinals.

Let $\mathbb{IM}$ be the class of functions which are both inflationary and monotone. ■

Recall that an ordinal ν is critical if $\nu = \omega^\nu$. These are sometimes rather quaintly referred to as ϵ-numbers. The five properties (i, si, m, sm, c) are standard. The two properties (b, sb) are not often named, but often used as a technical convenience. It doesn't take long to see that the three implications

$$si \Rightarrow i \qquad sm \Rightarrow i + m \qquad i + sb \Rightarrow b$$

hold. If g is continuous then

$$g\lambda = \bigvee\{g\alpha \mid \alpha < \lambda\}$$

for each limit ordinal λ. In general this is not enough to ensure continuity, but it is for monotone functions. Luckily we are concerned almost entirely with the class $\mathbb{IM}$ of functions.

In the next few paragraphs we obtain a few properties of $\mathbb{IM}$ and its members. Some of these properties hold for larger classes of ordinal functions, but we don't need those generalizations here.

As with any class of monotone functions the class $\mathbb{IM}$ can be partially ordered using the pointwise comparison.

$$f \leq g \Longleftrightarrow (\forall \alpha \in \mathbb{O}\mathrm{rd})[f\alpha \leq g\alpha]$$

With this comparison, for each non-empty countable subset $\mathcal{G} \subseteq \mathbb{IM}$ the pointwise supremum $\bigvee \mathcal{G}$ is the actual supremum.

3.2 LEMMA. *The class $\mathbb{IM}$ is smooth. Furthermore*

$$\beta \leq \alpha \Longrightarrow g^{\beta} \leq g^{\alpha}$$

for each $g \in \mathbb{IM}$ and ordinals α, β.

Proof. The first part is easy, and the second part follows by induction on α (making use of the inflationary property of g). ■

By intention each smooth class is closed under (non-zero) ordinal iterates. When $g \in \mathbb{IM}$ the family $\{g^{\alpha} \mid \alpha \in \mathbb{O}\mathrm{rd}\}$ of iterates is an ascending chain

$$id = g^0 \leq g = g^1 \leq g^2 \leq \cdots \leq g^{\alpha} \leq \cdots \quad (\alpha \in \mathbb{O}\mathrm{rd})$$

which helps with certain calculations. For instance, we have the following.

3.3 LEMMA. *If $g \in \mathbb{IM}$ then*

$$g^{\alpha} \circ g^{\beta} = g^{\beta+\alpha} \qquad (g^{\beta})^{\alpha} = g^{\beta\times\alpha}$$

for all $\alpha, \beta \in \mathbb{O}\mathrm{rd}$.

Proof. Both of these are proved by induction on α. Let's look at the leap to a limit ordinal λ for the second identity. Thus we require

$$(g^{\beta})^{\lambda}\zeta = g^{\beta\times\lambda}\zeta$$

for each $\zeta \in \mathbb{O}\mathrm{rd}$. We have

$$(g^{\beta})^{\lambda}\zeta = \bigvee\{(g^{\beta})^{\alpha}\zeta \mid \alpha < \lambda\} = \bigvee\{g^{\beta\times\alpha}\zeta \mid \alpha < \lambda\} \qquad g^{\beta\times\lambda}\zeta = \bigvee\{g^{\gamma}\zeta \mid \gamma < \beta\times\lambda\}$$

where the second equality uses the induction hypothesis. Also

$$\beta \times \lambda = \bigvee\{\beta \times \alpha \mid \alpha < \lambda\}$$

(by construction of ordinal multiplication). The comparison

$$(g^{\beta})^{\lambda}\zeta \leq g^{\beta\times\lambda}\zeta$$

is immediate (since $\beta \times \alpha \leq \beta$ for $\alpha < \lambda$). For the converse consider any $\gamma < \beta$. There is some $\alpha < \lambda$ with $\gamma \leq \beta$, and then Lemma 3.2 gives

$$g^{\gamma}\zeta \leq g^{\beta\times\alpha}\zeta \leq (g^{\beta})^{\lambda}\zeta$$

which leads to the required result. ■

We have made a bit of a meal of this proof to highlight the required properties of g.

As a consequence of this result some limit iterates of functions in $\mathbb{IM}$ are constant for long periods, and so can not be strictly monotone.

3.4 EXAMPLE. Let $g \in \mathbb{IM}$ and suppose λ is additively critical. For each ordinal ζ and ordinal $\alpha < \lambda$ we have

$$g^{\lambda}(g^{\alpha}\zeta) = (g^{\lambda} \circ g^{\alpha})\zeta = g^{\alpha+\lambda}\zeta = g^{\lambda}\zeta$$

and hence g^{λ} is constant between ζ and $g^{\alpha}\zeta$.

When g is continuous the situation is even more dramatic. Suppose g is continuous and observe that each ordinal iterate g^{α} is continuous, In particular, g^{λ} is continuous. Thus

$$g^{\lambda\cdot 2}\zeta = g^{\lambda}(g^{\lambda}\zeta) = g^{\lambda}\left(\bigvee\{g^{\alpha}\zeta \mid \alpha < \zeta\}\right) = \bigvee\{g^{\lambda}(g^{\alpha}\zeta) \mid \alpha < \zeta\} = g\zeta$$

and then for each $r < \omega$ we obtain

$$g^{\lambda\cdot(r+1)}\zeta = g^{\lambda}\zeta$$

by a simple induction on r. Thus $g^{\lambda} = g^{\lambda\cdot\omega}$, and this indicates that there can be even longer constant stretches. ■

Finally, we need a couple of simple properties of $\mathbb{IM}$.

3.5 LEMMA. *For all function $f, g \in \mathbb{IM}$ we have*

$$g \leq g^2 \qquad f \leq g \Longrightarrow f^{\alpha} \leq g^{\alpha}$$

for each ordinal α for the right hand implication.

Proof. Consider any ordinal $\zeta \in \mathbb{O}\mathrm{rd}$. We have

$$\zeta \leq g\zeta$$

since g is inflationary, and then

$$g\zeta \leq g(g\zeta) = g^2\zeta$$

since g is monotone.

Suppose $f \leq g$ where $f, g \in \mathbb{IM}$. We show

$$f^{\alpha} \leq g^{\alpha}$$

by induction on α.

The base case, $\alpha = 0$, is immediate.

For the induction step, $\alpha \mapsto \alpha + 1$, we have

$$f^{\alpha+1} = f \circ f^{\alpha} \le f \circ g^{\alpha} \le g \circ g^{\alpha} = g^{\alpha+1}$$

where the first comparison holds by the induction hypothesis and since f is monotone, and the second comparison holds since $f \le g$.

For a limit ordinal λ we have

$$f^{\lambda} = \bigvee\{f^{\alpha} \mid \alpha < \lambda\} \le \bigvee\{g^{\alpha} \mid \alpha < \lambda\} = g^{\lambda}$$

to complete the induction. ■

We can now isolate the fruitful and the helpful functions.

Recall that an ordinal function $f \in \mathbb{O}\mathrm{rd}'$ is normal if it is strictly monotone and continuous. We modify this notion slightly.

3.6 DEFINITION. An ordinal function $f \in \mathbb{O}\mathrm{rd}'$ is fruitful if it is inflationary, monotone, big, and continuous. Let $\mathbb{F}\mathrm{ruit}$ be the class of fruitful functions.

An ordinal function $g \in \mathbb{O}\mathrm{rd}'$ is big normal if it is strictly monotone, big, and continuous.

An ordinal function $h \in \mathbb{O}\mathrm{rd}'$ is helpful if it is strictly inflationary, monotone, and strictly big. Let $\mathbb{H}\mathrm{elp}$ be the class of helpful functions. ■

Of course, these big normal functions are just the usual normal functions that are big. These are the only normal functions we need here so we often omit the modifier 'big'.

It turns out that *strict* monotonicity is rather too restrictive, so we use the larger class $\mathbb{F}\mathrm{ruit}$ of fruitful functions. Each such function g belongs to $\mathbb{IM}$, and Example 3.4 shows that not all the iterates are normal. We rectify that by releasing our grip on normality and becoming fruity.

A proof of the following is straight forward.

3.7 LEMMA. *Each of the classes* $\mathbb{F}\mathrm{ruit}$ *and* $\mathbb{H}\mathrm{elp}$ *is smooth.*

Why are the classes $\mathbb{F}\mathrm{ruit}$ and $\mathbb{H}\mathrm{elp}$ useful? To answer that we introduce a particular second level function.

3.8 DEFINITION. Let $\boldsymbol{Fix} : \mathbb{O}\mathrm{rd}''$ be the function given by

$$\boldsymbol{Fix} f\zeta = f^{\omega}(\zeta + 1)$$

for each function $f : \mathbb{O}\mathrm{rd}'$ and ordinal ζ. ■

We have

$$\boldsymbol{Fix} f\zeta = \bigvee\{f^{r}(\zeta + 1) \mid r < \omega\}$$

and this makes sense for any function $f : \mathbb{O}\mathrm{rd}'$. However, we use $\boldsymbol{Fix}$ only on $f \in \mathbb{F}\mathrm{ruit}$. For such f we see that $\boldsymbol{Fix}$ is a fixed point extractor.

3.9 LEMMA. *For each* $f \in \mathbb{F}\mathrm{ruit}$ *and* $\zeta \in \mathbb{O}\mathrm{rd}$*, the value* $\boldsymbol{Fix} f\zeta$ *is the least ordinal* ν *such that* $\zeta < \nu = f\nu$*. Furthermore, this value* ν *is critical.*

Proof. For the given $f \in \mathbb{F}\mathrm{ruit}$ and ordinal ζ, let $\nu = \boldsymbol{Fix} f\zeta$. Let

$$\zeta[r] = f^r(\zeta + 1)$$

for $r < \omega$, so that

$$\zeta < \zeta[0] \leq \cdots \leq \zeta[r] \leq \cdots$$

since f is inflationary and monotone, with

$$\nu = \bigvee\{\zeta[r] \mid r < \omega\}$$

by unravelling the definition of $\boldsymbol{Fix}$. Since f is continuous this gives

$$f\nu = \bigvee\{f\zeta[r] \mid r < \omega\} = \bigvee\{\zeta[r+1] \mid r < \omega\} = \nu$$

to show that ν is a fixed point of f.

Let μ be any fixed point of f with $\zeta < \mu$. Then $\zeta[0] \leq \mu$ and hence since f is monotone a simple induction gives

$$\zeta[r] \leq f\mu = \mu$$

for each $r < \omega$. Thus $\nu \leq \mu$.

Finally, since f is big we have

$$\zeta < \nu \leq \omega^\nu \leq f\nu = \nu$$

to show that ν is critical. ■

Much of the standard material on ordinal notations is about extracting fixed points, so we can see why $\boldsymbol{Fix}$ might be useful.

Fruitful and helpful functions work hand in hand.

3.10 LEMMA. *(a) For each $f \in \mathbb{F}\mathrm{ruit}$ the function $\boldsymbol{Fix} f$ is helpful.*

(b) For each $h \in \mathbb{H}\mathrm{elp}$ and ordinal ζ, the ordinal function $\alpha \mapsto h^\alpha\zeta$ is normal.

Proof. (a) For fruitful f let $h = \boldsymbol{Fix} f$. For $\zeta \in \mathbb{O}\mathrm{rd}$ let $\nu = h\zeta$. By Lemma 3.9, we have $\zeta < \nu = f\nu$ with a certain minimality on ν. In particular, h is strictly inflationary.

Consider any $\zeta \leq \eta$ and let $\mu = h\eta$. Then $\zeta \leq \eta < \mu = f\mu$ and hence $\nu \leq \mu$ by the minimality of ν. This shows that h is monotone.

Since $\nu \neq 0$, we have $\omega^\nu \leq f\nu = \nu$, so that ν is critical, and hence h is strictly big.

(b) For the given helpful function h and ordinal ζ let

$$f\alpha = h^\alpha\zeta$$

for each $\alpha \in \mathbb{O}\mathrm{rd}$.

By construction for each ordinal α we have

$$f(\alpha + 1) = h(f\alpha) > f\alpha$$

since h is strictly inflationary. Also by construction we have

$$f\lambda = \bigvee\{f\alpha \mid \alpha < \lambda\}$$

for each limit ordinal λ. But if $\alpha < \lambda$ then $\alpha < \alpha + 1 < \lambda$ so that $f\alpha < f(\alpha + 1) \leq f\lambda$ which is enough to show that f is strictly monotone.

A simple argument now shows that f is continuous.

For non-zero α the value $f\alpha$ is either a value of h or a supremum of such values. Thus $f\alpha$ is critical. But $\alpha \leq f\alpha$ and hence $\omega^\alpha \leq \omega^{f\alpha} = f\alpha$ to show that f is big. ■

By part (b) of this result, for each helpful function h and ordinal ζ the function $\alpha \mapsto h^\alpha\zeta$ is fruitful (in fact, normal) and provides an enumeration of a set of critical ordinals. This will be useful if only we can find some helpful functions. That is where part (a) comes into play.

The smallest fruitful function is $\omega^\bullet$, exponentiation to base ω.

3.11 DEFINITION. Let $\boldsymbol{Next} = \boldsymbol{Fix}(\omega^\bullet)$. ■

By Lemma 3.10(a) the function $\boldsymbol{Next}$ is helpful. Let h be any helpful function, and let $f = \omega^\bullet$. For each $\zeta \in \mathbb{O}\mathrm{rd}$ we have $\zeta+1 \leq h\zeta$ and $h\zeta$ is critical, so that $f(\zeta+1) \leq \omega^{h\zeta} = h\zeta$. An easy induction gives $f^r(\zeta + 1) \leq h\zeta$ for all $r \leq \omega$, and hence $\boldsymbol{Next}\zeta \leq h\zeta$, to show the following

3.12 LEMMA. *The function* $\boldsymbol{Next}$ *is the smallest helpful function.*

A simple argument shows that $\boldsymbol{Next}\,\zeta$ is the next critical ordinal strictly beyond ζ. In particular

$$\epsilon_\alpha = \boldsymbol{Next}^\alpha\epsilon_0 = \boldsymbol{Next}^{1+\alpha}\omega = \boldsymbol{Next}^{1+\alpha}0$$

is a long list of critical ordinals.

Suppose we try to use this iteration to generate notations from below. Once we have generated a critical ordinal ϵ_α then the Cantor normal form gives us a notation for all ordinals $\zeta < \epsilon_{\alpha+1}$. However, the rule is that before we can use ϵ_α we must have generated earlier a notation for α. Thus this process will run out of steam at

$$\epsilon_{\epsilon_{\epsilon_{\cdot_{\cdot}}}}$$

the least ordinal ν with $\epsilon_\nu = \nu$.

To generate larger critical ordinals we need more powerful helpful functions. We show how to produce these in the next section. To conclude this section we obtain a couple of properties of an arbitrary helpful function.

3.13 LEMMA. *Suppose* $h \in \mathbb{H}\mathrm{elp}$ *and let* λ *be an (additively) critical ordinal. Then for all ordinals* α, ν, ζ *with* $\zeta < \lambda$*, the three conditions*

(a) $\zeta + \alpha \leq h^\alpha\zeta$.

(b) $h^\lambda\zeta = h^\lambda 0$

(c) $(\zeta < \nu = h^\nu 0) \Longleftrightarrow (0 < \nu = h^\nu\zeta)$

hold.

Proof. (a) We prove this by induction on α.

The base case, $\alpha = 0$, is trivial.

For the induction step, $\alpha \mapsto \alpha + 1$, since h is strictly inflationary we have

$$h^{\alpha+1}\zeta = h(h^{\alpha}\zeta) \geq h^{\alpha}\zeta + 1 \geq \zeta + \alpha + 1$$

using the induction hypothesis.

For the induction leap to a limit ordinal λ we have

$$h^{\lambda}\zeta = \bigvee\{h^{\alpha}\zeta \,|\, \alpha < \lambda\} \geq \bigvee\{\zeta + \alpha \,|\, \alpha < \lambda\} = \zeta + \bigvee\{\alpha \,|\, \alpha < \lambda\} = \zeta + \lambda$$

as required.

(b) The iterate h^{λ} is helpful, and hence monotone, so that $h^{\lambda}\zeta \geq h^{\lambda}0$. For the converse we have $\zeta \leq h^{\zeta}0$ (by part (a)) and hence $h^{\lambda}\zeta \leq h^{\lambda}(h^{\zeta}0) = h^{\zeta+\lambda}0$ by Lemma 3.3. But $\zeta < \lambda$ and λ is (additively) critical so that $\zeta + \lambda = \lambda$, to give the required result.

(c) As a preliminary we observe that for each pair ν, ζ of ordinals with $\nu \neq 0$, the ordinal

$$\mu = h^{\nu}\zeta$$

is critical. If $\nu = \alpha + 1$ then

$$\mu = h(h^{\alpha}\zeta)$$

so that μ is critical since it is a value of the helpful function h. If ν is a limit ordinal, then μ is a supremum of values of h, and so is again critical.

As a particular case, if $\nu > 0$ and either of

$$\nu = h^{\nu}0 \qquad \nu = h^{\nu}\zeta$$

then ν is critical. Furthermore, if the right hand equality holds then $\zeta < \nu$. Two uses of (b) now gives the equivalence. ∎

A standard development of ordinal notations would make much use of normal functions. We will see that fruitful functions are a more amenable way of doing this. The helpful functions provide a canonical way of generating such functions. However, a more important benefit of the notion of helpfulness is that it lifts to higher levels.

4 Helpful functions

By Section 3 we know that for each helpful function $h \in \mathbb{H}\mathrm{elp}$ the fruitful function

$$\alpha \mapsto h^{1+\alpha}0$$

generates a sequence of critical ordinals. (The '1+' can be avoided here if we are prepared to start from a known critical ordinal.) Furthermore, for each fruitful function $f \in \mathbb{F}\mathrm{ruit}$, the function $\boldsymbol{Fix} f$ is helpful, and

$$\alpha \mapsto (\boldsymbol{Fix} f)^{1+\alpha}0$$

enumerates the fixed points of f.

In this section we describe a method of producing helpful functions which doesn't require a given fruitful function. The technique makes use of functions $g : \mathbb{O}\mathrm{rd}^{(l+1)}$ on all levels.

Recall that by decomposing the space

$$\mathbb{O}\mathrm{rd}^{(l+2)} = \mathbb{O}\mathrm{rd}^{(l+1)} \to \mathbb{O}\mathrm{rd}^{(l)} \to \cdots \to \mathbb{O}\mathrm{rd}' \to \mathbb{O}\mathrm{rd} \to \mathbb{O}\mathrm{rd}$$

we see that each function $H : \mathbb{O}\mathrm{rd}^{(l+2)}$ must receive successive inputs

$$h : \mathbb{O}\mathrm{rd}^{(l+1)},\ h_l : \mathbb{O}\mathrm{rd}^{(l)},\ \ldots,\ h_1 : \mathbb{O}\mathrm{rd}',\ \zeta : \mathbb{O}\mathrm{rd}$$

to return its eventual output $Hhh_l \cdots h_1\zeta$. Often these central inputs $h_l, \ldots, h_1$ play only a passive role, so we abbreviate the list $h_l \cdots h_1$ to $\boldsymbol{h}$ and write $Hh\boldsymbol{h}\zeta$ for the eventual output. We do not use this abbreviation in the following definition, but we will in the subsequent analysis.

4.1 DEFINITION. (Base) Let $\mathbb{H}^{(1)} = \mathbb{H}\mathrm{elp}$, the class of helpful functions on level 1.

(Step) For each $l < \omega$ a function $H : \mathbb{O}\mathrm{rd}^{(l+2)}$ is helpful on level $l+2$ if

$$\begin{array}{ll} \text{(Help1)} & Hh \text{ is helpful on level } l+1 \\ \text{(Help2)} & h^2 h_l \cdots h_1 \le Hhh_l \cdots h_1 \\ \text{(Help3)} & Hhh_l \cdots h_2 f \le Hhh_l \cdots h_2 g \end{array}$$

for all $h : \mathbb{H}^{(l+1)}$, $h_l : \mathbb{H}^{(l)}$, $\ldots$, $h_1 : \mathbb{H}^{(1)}$, and $f, g : \mathbb{H}^{(1)}$ with $f \le g$.

Let $\mathbb{H}^{(l+2)}$ be the class of helpful functions on level $l+2$. ■

This is a construction by recursion on the level l to produce $\mathbb{H}^{(l+1)} \subseteq \mathbb{O}\mathrm{rd}^{(l+1)}$. The comparison in (Help2, Help3) takes place in $\mathbb{O}\mathrm{rd}'$. Since the functions involved are in $\mathbb{IM}$, this doesn't lead to difficulties. Notice that the first two defining clauses of $\mathbb{H}^{(l+2)}$ can be written

$$\text{(Help1)} \quad Hh \in \mathbb{H}^{(l+1)} \qquad \text{(Help2)} \quad h^2\boldsymbol{h} \le Hh\boldsymbol{h}$$

using the abbreviation $\boldsymbol{h}$ explained above. The third clause is not so straight forward. In particular, for the case $l = 0$ you should read (Help3) with some care, because the sequence $h, h_l, \ldots, h_2$ is empty. A function $H : \mathbb{O}\mathrm{rd}''$ is in $\mathbb{H}^{(2)}$ precisely when

$$(1) \quad Hh \in \mathbb{H}^{(1)} \qquad (2) \quad h^2 \le Hh \qquad (3) \quad Hf \le Hg$$

for all $f, g, h : \mathbb{H}^{(1)}$ with $f \le g$.

The squaring property (Help2) is quite powerful, especially when used at higher levels. This will be a crucial component of the proof of several results.

4.2 LEMMA. *For each $H \in \mathbb{H}^{(l+2)}$, $h \in \mathbb{H}^{(l+1)}$, $h_l \in \mathbb{H}^{(l)}$, $\ldots$, $h_1 \in \mathbb{H}^{(1)}$ we have*

$$(h\boldsymbol{h})^2 \le Hh\boldsymbol{h}$$

(where $\boldsymbol{h}$ abbreviates $h_l \cdots h_1$).

Proof. We proceed by induction on the level l. For the base case, $l = 0$, the parameter sequence $\boldsymbol{h}$ is empty, and the required comparison $h^2 \leq Hh$ is just (Help2). For the induction step, $l \mapsto l+1$, consider a helpful $K : \mathbb{O}\mathrm{rd}^{(l+3)}$, as well as the helpful $H, h, \boldsymbol{h}$. By (Help1) we know that Hh is helpful. Thus, using (Help2) for K and the induction hypothesis, we have

$$KHh\boldsymbol{h} \geq H^2h\boldsymbol{h} = H(Hh)\boldsymbol{h} \geq (Hh\boldsymbol{h})^2$$

as required. ■

This result gives us another squaring property.

4.3 LEMMA. *Consider any list*

$$h_{l+1} \in \mathbb{H}^{(l+1)},\ h_l \in \mathbb{H}^{(l)},\ \ldots,\ h_1 \in \mathbb{H}^{(1)}$$

of helpful functions on the indicated levels. Then for each $1 \leq m \leq l$ the pointwise comparison

$$h_m^2 h_{m-1} \cdots h_1 \leq h_{l+1}h_l \cdots h_1$$

holds.

Proof. Since h_{m+1} is helpful we have

$$h_m^2 h_{m-1} \cdots h_1 \leq h_{m+1}h_m \cdots h_1$$

by (Help2). This is the required result if $m = l$. Otherwise Lemma 3.5 gives

$$h_{m+1}h_m \cdots h_1 \leq (h_{m+1}h_m \cdots h_1)^2$$

and then Lemma 4.2 gives

$$h_m^2 h_{m-1} \cdots h_1 \leq h_{m+2}h_{m+1}h_m \cdots h_1$$

since h_{m+2} is helpful. Repeating this argument eventually gives the required result. ■

In Lemma 3.7 we saw that the class $\mathbb{H}^{(1)} = \mathbb{H}\mathrm{elp}$ is smooth. We now generalize this.

4.4 LEMMA. *For each $l < \omega$, the class $\mathbb{H}^{(l+1)}$ is smooth.*

Proof. Lemma 3.7 gives the result for $\mathbb{H}^{(1)}$. We look at $\mathbb{H}^{(l+2)}$ for arbitrary $l < \omega$.

We show first that $\mathbb{H}^{(l+2)}$ is closed under composition. To this end consider any $G, H \in \mathbb{H}^{(l+2)}$. To show $G \circ H \in \mathbb{H}^{(l+2)}$ we look at (Help1, Help2, Help3) in turn.

For each $h \in \mathbb{H}^{(l+1)}$ we have $Hh \in \mathbb{H}^{(l+1)}$ and hence $G(Hh) \in \mathbb{H}^{(l+1)}$ to verify (Help1).

To verify (Help2) consider any compatible family $h, \boldsymbol{h}$ of helpful functions. Then

$$(G \circ H)h\boldsymbol{h} = G(Hh)\boldsymbol{h} \geq (Hh\boldsymbol{h})^2 \geq Hh\boldsymbol{h} \geq h^2\boldsymbol{h}$$

as required. Here the first comparison follows by Lemma 4.2, the second follows since $Hh\boldsymbol{h}$ is inflationary, and the third uses (Help2) for H.

To verify (Help3) observe that

$$(G \circ H)hh_l \cdots h_2 f = G(Hh)h_l \cdots h_2 f \qquad (G \circ H)hh_l \cdots h_2 g = G(Hh)h_l \cdots h_2 g$$

so the known monotone property of G gives the required result. (Strictly speaking, this is the argument for $l \neq 0$. A slight variant is needed for $l = 0$.)

To show that $\mathbb{H}^{(l+2)}$ is closed under pointwise suprema, consider a non-empty subset $\mathcal{H}$ of $\mathbb{H}^{(l+2)}$. We show that $\bigvee \mathcal{H}$ is helpful. We look at (Help1, Help2, Help3) in turn.

For each $h \in \mathbb{H}^{(l+1)}$ we have

$$(\bigvee \mathcal{H})h = \bigvee\{Hh \mid H \in \mathcal{H}\}$$

so the known closure property of $\mathbb{H}^{(l+1)}$ gives the required property of $\mathbb{H}^{(l+2)}$. (Strictly speaking, this is a proof by induction on l.) This verifies (Help1).

To verify (Help2) consider any compatible family $h, \boldsymbol{h}$ of helpful functions. Since $\mathcal{H}$ is non-empty we have

$$(\bigvee \mathcal{H})h\boldsymbol{h} \geq Hh\boldsymbol{h} \geq h^2\boldsymbol{h}$$

for any selected any member H of $\mathcal{H}$.

Property (Help3) follows in the same way. ■

For each $l < \omega$ the class $\mathbb{H}^{(l+2)}$ is smooth, and hence is closed under ordinal iteration. This has a useful consequence.

4.5 COROLLARY. *For each $l < \omega, H \in \mathbb{H}^{(l+2)}, h \in \mathbb{H}^{(l+1)}$ we have $H^\alpha h \in \mathbb{H}^{(l+1)}$ for each $\alpha \in \mathbb{O}\mathrm{rd}$.*

Lemmas 3.10(b) and 3.13 give us some crucial properties of helpful functions $h \in \mathbb{H}^{(1)}$. These properties lift to higher levels.

4.6 LEMMA. *Let $H : \mathbb{H}^{(l+2)}$, $h : \mathbb{H}^{(l+1)}$, $h_l : \mathbb{H}^{(l)}$, ..., $h_1 : \mathbb{H}^{(1)}$. Then the function*

$$\alpha \mapsto H^\alpha h\boldsymbol{h}\zeta$$

is normal, and

(a) $H^\alpha h\boldsymbol{h}\alpha \leq H^{\alpha+1}h\boldsymbol{h}0$

(b) $H^\lambda h\boldsymbol{h}\zeta = H^\lambda h\boldsymbol{h}0$

(c) $(\zeta < \nu = H^\nu h\boldsymbol{h}0) \iff (0 < \nu = H^\nu h\boldsymbol{h}\zeta)$

hold for all ordinals α, ν, ζ and (additively) critical ordinal λ with $\zeta < \lambda$. Here $\boldsymbol{h}$ abbreviates $h_l \cdots h_1$.

Proof. Let f be this function, that is

$$f\alpha = H^\alpha h\boldsymbol{h}\zeta$$

for each $\alpha \in \mathbb{O}\mathrm{rd}$.

The function $h\boldsymbol{h}$ is helpful, hence strictly inflationary, so that

$$Hh\boldsymbol{h}\zeta \geq (h\boldsymbol{h})^2\zeta = h\boldsymbol{h}(h\boldsymbol{h}\zeta) > h\boldsymbol{h}\zeta$$

by Lemma 4.2. In particular

$$f(\alpha+1) = H(H^\alpha h)\boldsymbol{h}\zeta > H^\alpha h\boldsymbol{h}\zeta = f\alpha$$

(using $H^\alpha h$ in place of h).

For each limit ordinal λ and ordinal $\alpha < \lambda$, we have

$$\alpha + 1 < \lambda$$

and so (by the definition of $f\lambda$) we have $f\alpha < f(\alpha+1) \leq f\lambda$ using the previous observation.

This shows that f is strictly monotone.

By construction the function f is continuous.

Finally, for each α the function $H^\alpha h\boldsymbol{h}$ is helpful, and so takes only critical values. But $\alpha \leq f\alpha$, so that $\omega^\alpha \leq \omega^{f\alpha} = f\alpha$, as required to show that f is normal.

(a) Using Lemma 4.2 we have

$$H^{\alpha+1}h\boldsymbol{h}0 = H(H^\alpha h)\boldsymbol{h}0 \geq (H^\alpha h\boldsymbol{h})^2 0 = H^\alpha h\boldsymbol{h}(H^\alpha h\boldsymbol{h}0) \geq H^\alpha h\boldsymbol{h}\alpha$$

where last comparison holds since each helpful function is inflationary.

(b) The comparison $H^\lambda h\boldsymbol{h}\zeta \geq H^\lambda h\boldsymbol{h}0$ is immediate.

For the converse consider any ordinal α with $\zeta < \alpha < \lambda$. Then, using part (a) we have

$$H^\alpha h\boldsymbol{h}\zeta \leq H^\alpha h\boldsymbol{h}\alpha \leq H^{\alpha+1}h\boldsymbol{h}0 \leq H^\lambda h\boldsymbol{h}0$$

where the last comparison holds by the construction of H^λ. Thus taking the supremum over all $\alpha < \lambda$ gives the required result.

(c) For the given $H, h, \boldsymbol{h}$ let

$$\mu = H^\nu h\boldsymbol{h}\zeta$$

for any pair ν, ζ of ordinals with $\nu \neq 0$. By Corollary 4.5 we see that μ is a value of a member of $\mathbb{H}^{(1)}$, and hence is critical. Note also that $\zeta < \mu$, so that two uses of part (b) gives

$$\mu = H^\nu h\boldsymbol{h}\eta$$

for each $\eta < \nu$. Further uses of part (b) give the two required implications. ■

That's enough of the generalities. What we need now are some useful examples of helpful functions on each level.

5 The higher level fixed point extractors

Lemma 3.10(a) gives us $\boldsymbol{Fix} f \in \mathbb{H}^{(1)}$ for each $f \in \mathbb{F}\mathrm{ruit}$. The aim of this section is to exhibit an important member $[\,\imath\,]$ of $\mathbb{H}^{(l+2)}$ for each level l. Naturally, we begin with the definition.

5.1 DEFINITION. For each level l let $[\,\imath\,] : \mathbb{O}\mathrm{rd}^{(l+2)}$ be the function given by

$$[\,\imath\,]h\boldsymbol{h} = \boldsymbol{Fix} f \qquad \text{where } f\alpha = h^\alpha \boldsymbol{h}0 \text{ (for } \alpha \in \mathbb{O}\mathrm{rd})$$

for each compatible family $h, \boldsymbol{h}$ of (helpful) functions. ■

It is important to understand what these functions do, so let's take a look at $[\,0\,]$.

Consider any helpful function $h : \mathbb{O}\mathrm{rd}'$. By Lemma 3.10(b) we have a normal and therefore fruitful function $f : \mathbb{O}\mathrm{rd}'$ given by $f\alpha = h^\alpha 0$ (for $\alpha \in \mathbb{O}\mathrm{rd}$). By Lemmas 3.9 and 3.13(c), we have

$$[\,0\,]h\zeta = \big(\text{the least } \nu \text{ with } \zeta < \nu = h^\nu 0\big) = \big(\text{the least } \nu \text{ with } 0 < \nu = h^\nu \zeta\big)$$

for each $\zeta \in \mathbb{O}\mathrm{rd}$, and $\nu = h^\nu 0$ is critical.

5.2 LEMMA. *The operator*

$$[\,0\,] : \mathbb{O}\mathrm{rd}^{(2)}$$

is helpful, that is $[\,0\,] \in \mathbb{H}^{(2)}$.

Proof. We must show that

$$\text{(Help1)} \quad [\,0\,]h \in \mathbb{H}\mathrm{elp} \qquad \text{(Help2)} \quad h^2 \le [\,0\,]h \qquad \text{(Help3)} \quad [\,0\,]f \le [\,0\,]g$$

for all $f, g, h \in \mathbb{H}\mathrm{elp}$ with $f \le g$.

(Help1) Consider any $h \in \mathbb{H}\mathrm{elp}$. Using both parts of Lemma 3.10 we see first that the function

$$\alpha \longmapsto h^\alpha 0$$

is normal, and hence

$$[\,0\,]h = \boldsymbol{Fix}(\alpha \longmapsto h^\alpha 0)$$

is helpful.

(Help2) Consider any ordinal ζ and let

$$\nu = [\,0\,]h\zeta = h^\nu 0 = h^\nu \zeta$$

be the value at ζ. As above, we know that ν is critical, and so $2 \le \nu$. The function $\alpha \longmapsto h^\alpha \zeta$ is monotone, and hence

$$h^2\zeta \le h^\nu \zeta = \nu$$

to give the required result.

(Help3) Consider helpful functions $f \le g$. Consider any $\zeta \in \mathbb{O}\mathrm{rd}$, and let

$$\mu = [\,0\,]f\zeta \qquad \nu = [\,0\,]g\zeta$$

so that $\mu \le \nu$ is the required conclusion. By the characteristic property of $[\,0\,]$ given above we have

$$\zeta < \mu = f^\mu 0 \qquad \zeta < \nu = g^\nu 0$$

where both μ and ν are the least solution of that requirement.

By Lemma 3.10(b) the function $\alpha \longmapsto f^\alpha 0$ is inflationary (in fact, normal), and we have

$$f^\nu \le g^\nu$$

by Lemma 3.5. Thus

$$\nu \le f^\nu 0 \le g^\nu 0 = \nu$$

and hence $\mu \leq \nu$ by the minimality of μ. ■

In this section we generalize this argument to show that each function $[\imath]$ is helpful on the appropriate level. This will take a little longer, and the proof will, in part, subsume the argument above.

We use the characteristic property of $[\imath]$ which generalizes that of $[\mathsf{o}]$ given above.

By Lemma 3.9 and either Lemma 3.13(c) for $l = 0$ or Lemma 4.6(c) for $l > 0$ we obtain the following.

5.3 LEMMA. *For each level l and helpful functions*

$$h \in \mathbb{H}^{(l+1)},\ h_l \in \mathbb{H}^{(l)},\ \ldots,\ h_1 \in \mathbb{H}^{(1)}$$

we have

$$[\imath]h\boldsymbol{h}\zeta = (\textit{the least } \nu \textit{ with } \zeta < \nu = h^\nu \boldsymbol{h}0) = (\textit{the least } \nu \textit{ with } 0 < \nu = h^\nu \boldsymbol{h}\zeta)$$

for each ordinal $\zeta \in \mathbb{O}\mathrm{rd}$.

As before, here $\boldsymbol{h}$ abbreviates the list $h_l, \ldots, h_1$ of central inputs. We continue with this, with a couple of variations, and move with caution when $l = 0$.

We look at the three required properties (Help1, Help2, Help3) in reverse order.

5.4 LEMMA. *For each level l the operator*

$$[\imath] : \mathbb{O}\mathrm{rd}^{(l+2)}$$

has (Help3).

Proof. The case $l = 0$ is dealt with in the proof of Lemma 5.2. Thus here we may assume $l > 0$.

Consider helpful functions

$$h \in \mathbb{H}^{(l+1)},\ h_l \in \mathbb{H}^{(l)},\ \ldots,\ h_2 \in \mathbb{H}^{(2)} \quad \text{and} \quad f, g \in \mathbb{H}^{(1)}$$

with $f \leq g$. We are concerned with the functions

$$[\imath]hh_l \cdots h_2 f \qquad [\imath]hh_l \cdots h_2 g$$

which we abbreviate as

$$[\imath]h\boldsymbol{h}f \qquad [\imath]h\boldsymbol{h}g$$

and move with the usual care. Observe that since $l > 0$ we have here a slight variation on our usual convention. The list $\boldsymbol{h}$ is empty when $l = 1$.

Consider any $\zeta \in \mathbb{O}\mathrm{rd}$, and let

$$\mu = [\imath]h\boldsymbol{h}f\zeta \qquad \nu = [\imath]h\boldsymbol{h}g\zeta$$

so that $\mu \leq \nu$ is the required conclusion.

By the characteristic property of $[\imath]$ given above we have

$$\zeta < \mu = h^\mu \boldsymbol{h}f0 \qquad \zeta < \nu = h^\nu \boldsymbol{h}g0$$

where both μ and ν are the least solution of that requirement.

By Lemma 4.4 the class $\mathbb{H}^{(l+1)}$ is smooth, so we have

$$h^{\nu} \in \mathbb{H}^{(l+1)}$$

and hence

$$h^{\nu}\boldsymbol{h} \in \mathbb{H}^{(2)}$$

by a sequences of uses of (Help1). This gives

$$h^{\nu}\boldsymbol{h}f \le h^{\nu}\boldsymbol{h}g$$

by (Help3) at level 2.

By the first part of Lemma 4.6 the function

$$\alpha \longmapsto h^{\alpha}\boldsymbol{h}f0$$

is inflationary (in fact, normal). Thus

$$\zeta < \nu \le h^{\nu}\boldsymbol{h}f0 \le h^{\nu}\boldsymbol{h}g0$$

so that both

$$\zeta < \mu \le h^{\mu}\boldsymbol{h}f0 \qquad \zeta < \nu \le h^{\nu}\boldsymbol{h}f0$$

hold. The minimality of μ gives $\mu \le \nu$, as required. ■

Next we look at (Help2).

5.5 LEMMA. *For each level l the operator*

$$[\,\imath\,] : \mathbb{O}\mathrm{rd}^{(l+2)}$$

has (Help2).

Proof. By Lemma 5.2 we know that this holds for $l = 0$. Thus we may suppose that $l > 0$.

Consider helpful functions

$$h \in \mathbb{H}^{(l+1)},\ h_l \in \mathbb{H}^{(l)},\ \ldots,\ h_1 \in \mathbb{H}^{(1)}$$

and an ordinal $\zeta \in \mathbb{O}\mathrm{rd}$. We require

$$h^2\boldsymbol{h}\zeta \le [\,\imath\,]h\boldsymbol{h}\zeta$$

where $\boldsymbol{h}$ is the usual abbreviation.

Let

$$\nu = [\,1\,]h\boldsymbol{h}\zeta$$

so that

$$\zeta < \nu = h^{\nu}\boldsymbol{h}0 = h^{\nu}\boldsymbol{h}\zeta$$

by Lemma 5.3.

By the first part of Lemma 4.6 the function f given by

$$f\alpha = h^{\alpha}\boldsymbol{h}\zeta$$

is normal, and ν is a fixed point of the function. In particular, $2 < \nu$. Since f is monotone we have

$$h^{2}\boldsymbol{h}\zeta \le h^{\nu}\boldsymbol{h}\zeta = \nu$$

for the required result. ■

Verifying that $[\imath]$ has (Help1) will take a little longer.

We need another consequence of the squaring property of (Help2).

5.6 LEMMA. *For each level l and helpful functions*

$$h \in \mathbb{H}^{(l+1)},\ h_l \in \mathbb{H}^{(l)},\ \dots,\ h_1 \in \mathbb{H}^{(1)}$$

we have

$$h_m^2 h_{m-1} \cdots h_1 \le [\imath]h\boldsymbol{h}$$

for each $1 \le m \le l$.

Proof. Here

$$\boldsymbol{h} \quad \text{abbreviates} \quad h_l \cdots h_1$$

in the usual way.

Consider any $1 \le m \le l$. We have

$$h_m^2 h_{m-1} \cdots h_1 \le h h_l \cdots h_1 = h\boldsymbol{h}$$

by Lemma 4.3.

Consider any $\zeta \in \mathbb{O}\mathrm{rd}$. By Lemma 5.3 the value

$$\nu = [\imath]h\boldsymbol{h}\zeta$$

satisfies

$$0 < \nu = h^{\nu}\boldsymbol{h}\zeta$$

and is infinite (in fact critical). By Lemma 3.10 or the first part of Lemma 4.6 the function

$$\alpha \longmapsto h^{\alpha}\boldsymbol{h}\zeta$$

is monotone (in fact normal). Thus

$$h_m^2 h_{m-1} \cdots h_1\zeta \le h\boldsymbol{h}\zeta \le h^{\nu}\boldsymbol{h}\zeta = \nu$$

to give the required result. ■

How do we show that

$$[\imath] : \mathbb{O}\mathrm{rd}^{(l+2)}$$

has (Help1)? We must show that

$$[\imath]h \text{ is helpful}$$

for all $h : \mathbb{O}\mathrm{rd}^{(l+1)}$.

To do that we must show

(l-1) hh_l is helpful

(l-2) $[\,\imath\,]h$ has (Help2)

(l-3) $[\,\imath\,]h$ has (Help3)

for all $h \in \mathbb{H}^{(l+1)}, h_l \in \mathbb{H}^{(l)}$. Of these ($l2$) is the case $m = l$ of Lemma 5.6, and ($l3$) holds by Lemma 5.4. Thus is suffices to show (l-1).

To do that we must show

(($l-1$)-1) hh_lh_{l-1} is helpful

(($l-1$)-2) $[\,\imath\,]hh_l$ has (Help2)

(($l-1$)-3) $[\,\imath\,]hh_l$ has (Help3)

for all $h \in \mathbb{H}^{(l+1)}, h_l \in \mathbb{H}^{(l)}, h_{l-1} \in \mathbb{H}^{(l-1)}$. Of these ($l-12$) is the case $m = l-1$ of Lemma 5.6, and ($l3$) holds by Lemma 5.4. Thus is suffices to show (($l-1$)-1).

This indicates that to show that $[\,\imath\,]$ has (Help1) we must verify a list

$$(l\text{-}1) \quad ((l-1)\text{-}1) \quad \dots \quad (1\text{-}1)$$

of conditions. We do this by a restricted induction from 1 to l.

Here is the base case.

5.7 LEMMA. *For an arbitrary list*

$$h \in \mathbb{H}^{(l+1)},\ h_l \in \mathbb{H}^{(l)},\ \dots,\ h_1 \in \mathbb{H}^{(1)}$$

of helpful functions on the indicated levels, the function

$$[\,\imath\,]hh_l \cdots h_1 : \mathbb{O}\mathrm{rd}^{(1)}$$

is helpful.

Proof. With the usual abbreviation consider the function $f : \mathbb{O}\mathrm{rd}^{(1)}$ given by

$$f\alpha = h^\alpha \boldsymbol{h}0$$

for each ordinal α. By Lemma 3.10(b) for $l = 0$ or the first part of Lemma 4.6 for $l > 0$, this function f is normal. Thus

$$[\,\imath\,]h\boldsymbol{h} = \boldsymbol{Fix} f$$

is helpful by Lemma 3.10(a). ■

We build on this to obtain the following.

5.8 LEMMA. *Consider the fixed point extractor*

$$[\,\imath\,] : \mathbb{O}\mathrm{rd}^{(l+2)}$$

for an arbitrary level l. Consider also helpful functions

$$h \in \mathbb{H}^{(l+1)},\ h_l \in \mathbb{H}^{(l)},\ \dots,\ h_1 \in \mathbb{H}^{(1)}$$

on the indicated levels.

For each $1 \le m \le l$ the function

$$[\,\imath\,]hh_l \cdots h_m : \mathbb{O}\mathrm{rd}^{(m)}$$

is helpful, that is a member of $\mathbb{H}^{(m)}$.

Proof. We prove this by a restricted induction on m (from 1 to l) with allowable variations of the helpful functions $h, h_l, \dots, h_1$.

The base case, $m = 1$, is just Lemma 5.7.

Consider the induction step, $m \mapsto m + 1$, where $m < l$. To show that

$$[\imath]h_{l+1}h_l \cdots h_{m+1}$$

is helpful we require the following three conditions.

(1) $[\imath]h_{l+1}h_l \cdots h_{m+1}h_m$ is helpful

(2) $h_m^2 h_{m-1} \cdots h_m \le [\imath]h_{l+1}h_l \cdots h_{m+1}h_m \cdots h_1$

(3) $[\imath]h_{l+1}h_l \cdots h_{m+1}h_m \cdots h_2$ is 'monotone'

for all helpful functions $h_m, \dots, h_1$ on the appropriate levels.

Requirement (1) is the induction hypothesis.

Requirement (2) is is just Lemma 5.6.

Requirement (3) is a consequence of Lemma 5.4. ■

The preamble to Lemma 5.7 show that we have the following.

5.9 COROLLARY. *For each level l the operator*

$$[\imath] : \mathbb{O}\mathrm{rd}^{(l+2)}$$

has (Help1).

Finally, Corollary 5.9 and Lemmas 5.5 and 5.4 give the required result.

5.10 THEOREM. *For each level l the function*

$$[\imath] : \mathbb{O}\mathrm{rd}^{(l+2)}$$

is helpful, that is a member of $\mathbb{H}^{(l+2)}$

This fills the gaps of [7].

There is another possible development of this material.

The first observation is that a helpful function on some level is applied only to a helpful input at the next level down (where each ordinal is viewed as helpful). We may set $\mathbb{H}^{(0)} = \mathbb{O}\mathrm{rd}$. In fact, we are not interested in the behaviour of a helpful function outside the helpful inputs. The next observation is that a helpful function applied to a helpful input returns a helpful output. This is the condition (Help1). Thus we could define a

$$\mathbb{H}^{(l+1)} = \text{ those functions of type } \mathbb{H}^{(l)} \longrightarrow \mathbb{H}^{(l)}$$

which satisfy certain restriction. At the same time we can partially order each class $\mathbb{H}^{(l)}$. Since $\mathbb{H}^{(0)} = \mathbb{O}\mathrm{rd}$ we have the linear comparison on $\mathbb{H}^{(0)}$, and since $\mathbb{H}^{(1)} \subseteq \mathbb{I}\mathbb{M}$ we have the pointwise comparison on $\mathbb{H}^{(1)}$. This idea lifts all the way up the levels. Thus for $H, K \in \mathbb{H}^{(l+2)}$ we use

$$H \le K \iff (\forall h \in \mathbb{H}^{(l)})[Hh \le Kh]$$

to produce a pointwise comparison on $\mathbb{H}^{(l+2)}$. With this we find that

$$h^2 \leq Hh \qquad H \text{ is monotone}$$

are rephrasings of (Help2) and (Help3). Furthermore, the pointwise supremum on $\mathbb{H}^{(l)}$ is the actual supremum. As a consequence of this we obtain higher level analogues of Lemma 3.3. Thus

$$H^{\alpha} \circ H^{\beta} = H^{\beta \circ \alpha} \qquad (H^{\beta})^{\alpha} = H^{\beta \times \alpha}$$

holds for each $H \in \mathbb{H}^{(l+2)}$ and ordinals α, β.

The problem with this approach is that it requires the definitions and properties to be developed in parallel, which can be a bit messy. Thus I chose to present it as above.

6 Ascending the ordinals

In this final section I will indicate how the use of helpful functions relates to other methods of generating ordinal notations. Of course, [7] is concerned with the relationship with the modern method, but the earlier methods also fit into the same picture. I will indicate how this can be done without giving all the details. More information will be given in [8].

It is convenient to take an historical perspective. This enables us to produce a sequence

$$\Delta[0], \Delta[1], \Delta[2], \Delta[3], \ldots$$

of larger and larger ordinals the first few of which are milestones along the journey.

God created the natural numbers

$$0, 1, 2, \ldots$$

but, by design or oversight, forgot to tell us the limit point

$$\Delta[0] = \omega$$

of this sequence. This is the zeroth ordinal in our Δ-sequence. It was left to Cantor to discover ω and peer beyond. (Some people have tried to convince me that Cantor invented ω, not just discover it.)

It is fair to say that the Cantor normal form to base ω gives the first system of ordinal notations. This uses $\omega^{\bullet}$, exponentiation to base ω, and is good enough to name all the ordinals below ϵ_0, the least ordinal ϵ with $\omega^{\epsilon} = \epsilon$, the least critical ordinal. Thus this system closes off at

$$\Delta[1] = \boldsymbol{Next}\, \omega = \epsilon_0$$

the first ordinal in our sequence. Notice how this uses $\boldsymbol{Next}$, the simplest helpful function.

To go beyond ϵ_0 we must name larger critical ordinals. Since

$$\epsilon_{\alpha} = \boldsymbol{Next}^{1+\alpha}\omega = \boldsymbol{Next}^{1+\alpha}0$$

we can do this by iterating $\boldsymbol{Next}$. With these ordinals we can extend the Cantor normal form for quite a bit further. This extended system closes off at the least ordinal ν with $\nu = \boldsymbol{Next}^{\nu}\omega$, which is

$$\Delta[2] = [\,{\scriptstyle 0}\,]\boldsymbol{Next}\, \omega = \epsilon_{\epsilon_{\epsilon_{\epsilon_{\cdot_{\cdot}}}}}$$

the second ordinal in our sequence. This ordinal rarely gets a mention. I don't know what it has done to deserve that. I hope it is not some cardinal sin.

The next extension was made by Veblen. In [9] he constructed what we now call the Veblen hierarchy, and various iterated extensions of that. Let's try to understand what he did.

He introduced the notion (but not the terminology) of a normal function. He actually works with general normal functions but here, as in the rest of the paper, we may restrict our attention to big normal functions. His Theorem 4 shows that each normal function has arbitrarily large fixed points, and these fixed points are closed under countable suprema. In fact, in our notation, his proof introduces the second level function $\boldsymbol{Fix}$.

At the top of his page 284 he introduces the *first derived function* of a normal function f. This is the function f' that enumerates the fixed points of f, the function

$$\alpha \longmapsto \big(\boldsymbol{Fix} f\big)^{1+\alpha} 0$$

in our notation. His proof of his Theorem 4 shows that this function is normal.

In his Theorems 5 and 6 he shows that the process $f \mapsto f'$ can be iterated, and so produces an ordinal indexed hierarchy of normal functions. This is what we now call the Veblen hierarchy on the base f. Let's set this up in our notation.

We start from any normal function $f : \mathbb{O}\mathrm{rd}'$. In words we set

$$\begin{array}{ll} \phi[f]0 & = f \\ \phi[f](\alpha+1) & = \text{enumeration of fixed points of } \phi[f]\alpha \\ \phi[f]\lambda & = \text{enumeration of common fixed points of } \phi[f]\alpha \text{ for all } \alpha < \lambda \end{array}$$

for each ordinal α and limit ordinal λ. This gives us a whole hierarchy of normal functions with a substantial harvest of fixed points. With hindsight we see that we may generate such a hierarchy $\phi[f]$ on any fruitful function f, and this does have some simplifying consequences.

Let $\boldsymbol{Veb} : \mathbb{O}\mathrm{rd}''$ be function given by

$$\boldsymbol{Veb} f \zeta = h^{1+\zeta} 0 \quad \text{where } h = \boldsymbol{Fix} f$$

for $f : \mathbb{O}\mathrm{rd}'$ and $\zeta \in \mathbb{O}\mathrm{rd}$. Thus for each $f \in \mathbb{F}\mathrm{ruit}$ the function $\boldsymbol{Veb} f$ enumerates the fixed points of f. This second level function $\boldsymbol{Veb}$ can be iterated to produce a slightly different hierarchy

$$\alpha \longmapsto \boldsymbol{Veb}^{\alpha} f$$

of fruitful functions on any fruitful base function f. This is closely related to the Veblen hierarchy $\phi[f]$, but we have to remember that for each limit ordinal λ the accumulation level

$$\boldsymbol{Veb}^{\lambda} f$$

need not be normal (even if the base function f is normal).

To explain the precise connection we need an observation about directed families of fruitful functions.

6.1 LEMMA. *For each countable directed family $\mathcal{F}$ of fruitful functions we have*

$$\boldsymbol{Fix}(\bigvee \mathcal{F}) = \bigvee \boldsymbol{Fix}[\mathcal{F}]$$

and the common fixed points of the members of $\mathcal{F}$ are the fixed points of the function $\bigvee \mathcal{F}$.

As a particular case of this, for each limit ordinal λ the common fixed points of the family

$$\{\,\boldsymbol{Veb}^{\alpha}f \mid \alpha < \lambda\}$$

are precisely the fixed points of the function

$$\boldsymbol{Veb}^{\lambda}f = \bigvee\{\,\boldsymbol{Veb}^{\alpha}f \mid \alpha < \lambda\}$$

to give the following.

6.2 LEMMA. *We have*

$$\phi[f](1+\alpha) = \boldsymbol{Veb}^{\alpha+1}f$$

for each fruitful function f and ordinal α.

This description of $\phi[f]$ omits the base function $f = \phi[f]0$ and each limit level of the iteration hierarchy. As we will see in a moment, each generated function $\phi[f](1+\alpha)$ is normal.

By Lemma 3.10 we have two function

$$\boldsymbol{Fix} : \mathbb{F}\mathrm{ruit} \longrightarrow \mathbb{H}\mathrm{elp} \qquad \boldsymbol{Enm} : \mathbb{H}\mathrm{elp} \longrightarrow \mathbb{F}\mathrm{ruit}$$

given by

$$\boldsymbol{Fix}\, f\zeta = \text{least } \nu \text{ with } \zeta < \nu = f\nu \qquad \boldsymbol{Enm}\, h\alpha = h^{1+\alpha}0$$

for each $f \in \mathbb{F}\mathrm{ruit}, h \in \mathbb{H}\mathrm{elp}$ and ordinals α, ζ. Recall also that $\boldsymbol{Enm}\, h$ is normal. These two functions can be composed both ways to produce functions we have seen already.

6.3 LEMMA. *Both*

$$\boldsymbol{Veb} = \boldsymbol{Enm} \circ \boldsymbol{Fix} \qquad [\,0\,] = \boldsymbol{Fix} \circ \boldsymbol{Enm}$$

hold.

Proof. The left hand equality is more or less the definition of $\boldsymbol{Veb}$. The verification of the right hand equality takes just a little longer.

For each helpful function h and ordinal ζ we have

$$\begin{aligned}[\,0\,]h\zeta &= \text{least } \nu \text{ with } \zeta < \nu = h^{\nu}0 \\ &= \text{least } \nu \text{ with } \zeta < \nu = h^{1+\nu}0 \\ &= \text{least } \nu \text{ with } \zeta < \nu = \boldsymbol{Enm}\, h\zeta \quad = \boldsymbol{Fix}(\boldsymbol{Enm}\, h)\zeta\end{aligned}$$

for the required result. ■

We now have

$$\boldsymbol{Fix} \circ \boldsymbol{Veb} = \boldsymbol{Fix} \circ \boldsymbol{Enm} \circ \boldsymbol{Fix} = [\,0\,] \circ \boldsymbol{Fix}$$

and this equality can be generalized.

6.4 LEMMA. *We have*

$$\boldsymbol{Fix} \circ \boldsymbol{Veb}^{\alpha} = [\,0\,]^{\alpha} \circ \boldsymbol{Fix}$$

for each ordinal α.

Proof. This follows by induction on α. The base case is trivial, the induction step follows by the observation above, and the induction leap to a limit ordinal follows by Lemma 6.1. ■

This result gives us a more 'helpful' description of the Veblen hierarchy.

6.5 THEOREM. *For each fruitful function f and ordinal α we have*

$$\phi[f](1+\alpha) = \boldsymbol{Enm}\big([\,0\,]^{\alpha}h\big)$$

where $h = \boldsymbol{Fix}f$.

Proof. Using Lemmas 6.2 and 6.4 we have

$$\begin{aligned}\phi[f](1+\alpha) &= \boldsymbol{Veb}^{\alpha+1}f\\ &= (\boldsymbol{Veb}\circ\boldsymbol{Veb}^{\alpha})f\\ &= (\boldsymbol{Enm}\circ\boldsymbol{Fix}\circ\boldsymbol{Veb}^{\alpha})f\\ &= (\boldsymbol{Enm}\circ[\,0\,]^{\alpha}\circ\boldsymbol{Fix})f \qquad = (\boldsymbol{Enm}\circ[\,0\,]^{\alpha})h\end{aligned}$$

for the required result ■

This hierarchy $\phi[f]$ has certain barrier ordinals, ordinals ν such that

$$\nu = \phi[f]\nu 0$$

holds. If we try to use $\phi[f]$ to generate ordinal notations from below then we can not get further than the least such barrier ordinal.

As an example consider the case $f = \omega^{\bullet}$. Then

$$h = \boldsymbol{Fix}f = \boldsymbol{Next}$$

and

$$\boldsymbol{Veb}f = \boldsymbol{Enm}(\boldsymbol{Next})$$

enumerates the critical ordinals. The barrier ordinals of $\phi[\omega^{\bullet}]$ are the strongly critical ordinals.

For an arbitrary fruitful function f what should we do to bring some of the barrier ordinals of $\phi[f]$ into the fold? Consider the function ${}^{\Uparrow}\!f$ given by

$${}^{\Uparrow}\!f\alpha = \phi[f](1+\alpha)0$$

for each ordinal α. This is a kind of diagonal limit through $\phi[f]$. Thus $\boldsymbol{Veb}\big({}^{\Uparrow}\!f\big)$ enumerates the barrier ordinals of $\phi[f]$.

Veblen shows that for each normal function f the associated function ${}^{\Uparrow}\!f$ is normal and suggests that we look at the hierarchy on this base. After that we iterate his procedure. That is the content of \$3 and \$4 of [9]. It is not easy reading, but we can now describe the process in a more compact way.

Remembering the form of $\boldsymbol{Enm}$ we see that ${}^{\Uparrow}\!f$ is given by

$${}^{\Uparrow}\!f\alpha = \phi[f](1+\alpha)0 = \boldsymbol{Enm}\big([\,0\,]^{\alpha}h\big)0 = [\,0\,]^{\alpha}h0$$

for each ordinal α. Here, of course, $h = \boldsymbol{Fix}f$. By the first part of Lemma 4.6 we immediately see that ${}^{\Uparrow}\!f$ is normal.

To generate $\phi[{}^{\Uparrow}\!f]$ in the form given by Theorem 6.5 we need to know what $\boldsymbol{Fix}\big({}^{\Uparrow}\!f\big)$ is. What can that be?

6.6 LEMMA. *For each fruitful function f we have*

$$\boldsymbol{Fix}\big({}^{\Uparrow}f\big) = [\,{\scriptstyle 1}\,][\,{\scriptstyle 0}\,]h$$

where $h = \boldsymbol{Fix}f$.

Proof. Remembering the characteristic properties of $\boldsymbol{Fix}$ and $[\,{\scriptstyle 1}\,]$ for each ordinal ζ we have

$$\boldsymbol{Fix}\big({}^{\Uparrow}f\big)\zeta = \big(\text{least } \nu \text{ with } \zeta < \nu = {}^{\Uparrow}f\nu = [\,{\scriptstyle 0}\,]^{\nu}h0\big) = [\,{\scriptstyle 1}\,][\,{\scriptstyle 0}\,]h\zeta$$

for the required result. ■

For the case $f = \omega^{\bullet}$ the Veblen hierarchy $\phi[\omega^{\bullet}]$ close off at

$$\Delta[3] = [\,{\scriptstyle 1}\,][\,{\scriptstyle 0}\,]\boldsymbol{Next}\omega = \Gamma_0$$

the third ordinal in our sequence.

To approach $\Delta[4]$ let's look at a few extensions of $\phi[f]$ in the form given by Theorem 6.5. With

$$h = \boldsymbol{Fix}f$$

we have

$$\begin{array}{ll}
\phi[f] & \alpha \longmapsto \boldsymbol{Enm}\Big([\,{\scriptstyle 0}\,]^{\alpha}h\Big) \\
\phi[{}^{\Uparrow}f] & \alpha \longmapsto \boldsymbol{Enm}\Big([\,{\scriptstyle 0}\,]^{\alpha}\big([\,{\scriptstyle 1}\,][\,{\scriptstyle 0}\,]h\big)\Big) \\
\phi[{}^{\Uparrow\Uparrow}f] & \alpha \longmapsto \boldsymbol{Enm}\Big([\,{\scriptstyle 0}\,]^{\alpha}\big(([\,{\scriptstyle 1}\,][\,{\scriptstyle 0}\,])^{2}h\big)\Big) \\
 & \qquad\vdots \\
\phi[{}^{(\beta)}f] & \alpha \longmapsto \boldsymbol{Enm}\Big([\,{\scriptstyle 0}\,]^{\alpha}\big(([\,{\scriptstyle 1}\,][\,{\scriptstyle 0}\,])^{\beta}h\big)\Big) \\
 & \qquad\vdots
\end{array}$$

where at the β^{th} stage we iterate the construction ${}^{\Uparrow}(\cdot)$ that number of ordinal times.

Where does this multi-hierarchy run out of steam, and what should we do then?

Consider the zeroth function

$$\phi[{}^{(\beta)}f]0$$

and consider the zeroth ordinal

$$\phi[{}^{(\beta)}f]00 = \boldsymbol{Enm}\Big(([\,{\scriptstyle 1}\,][\,{\scriptstyle 0}\,])^{\beta}h\Big)0 = ([\,{\scriptstyle 1}\,][\,{\scriptstyle 0}\,])^{\beta}h0$$

of that function. The function

$$\beta \longmapsto ([\,{\scriptstyle 1}\,][\,{\scriptstyle 0}\,])^{\beta}h0$$

is a kind of diagonal limit through the multi-hierarchy. Thus we should now generate the Veblen hierarchy on this base function, and then repeat our previous iteration.

By applying $\boldsymbol{Fix}$ to this function we get

$$[\,{\scriptstyle 1}\,]^{2}[\,{\scriptstyle 0}\,]h$$

and so we obtain a second multi-hierarchy

$$
\begin{gathered}
\alpha \longmapsto \boldsymbol{Enm}\Big([\,0\,]^{\alpha}\big([\,1\,]^{2}[\,0\,]h\big)\Big)\\
\alpha \longmapsto \boldsymbol{Enm}\Big([\,0\,]^{\alpha}\big(([\,1\,]^{2}[\,0\,])^{2}h\big)\Big)\\
\vdots\\
\alpha \longmapsto \boldsymbol{Enm}\Big([\,0\,]^{\alpha}\big(([\,1\,]^{2}[\,0\,])^{\beta}h\big)\Big)\\
\vdots
\end{gathered}
$$

which extends the previous one.

We are beginning to see some structure here. For each helpful $H \in \mathbb{H}^{(2)}$ we have a hierarchy of functions

$$\alpha \longmapsto \boldsymbol{Enm}\big([\,0\,]^{\alpha}(Hh)\big)$$

determined by H. Thus if we can find a way of passing through $\mathbb{H}^{(2)}$ then we can produce a large family of hierarchies. This is what Veblen's \$3 and \$4 are about although, of course, he didn't explain it in this way.

Veblen described an intricate system of indexing which, with hindsight, can be seen to produce a large collection of members of $\mathbb{H}^{(2)}$. I think it is fair to say that by that stage Veblen's description is not exactly crystal clear.

In [6] Schütte reorganized Veblen's method to produce an exotic, not to say eccentric, method of indexing many layers of Veblen hierarchies. He produced certain arrays of ordinals, Schütte brackets, which interact with a normal function to produce an ordinal. Using helpful functions in general, and the two functions $[\,0\,]$ and $[\,1\,]$ in particular, it is possible to give an explicit description of each bracket without going through any kind of recursion.

Looking again at the various hierarchies generated above, we see there is a rather simple way of passing through all of them. The function

$$\alpha \longmapsto \boldsymbol{Enm}\big([\,1\,]^{\alpha}[\,0\,]h\big)$$

is a kind of diagonal limit through the whole battery of hierarchies. For the particular case $f = \omega^{\bullet}$ the least barrier ordinal of this hierarchy is

$$\Delta[4] = [\,2\,][\,1\,][\,0\,]\boldsymbol{Next}\,\omega$$

the next ordinal in our sequence. This is sometimes known as the Ackermann ordinal. It is the upper bound of the ordinals that can be named by the methods developed by Veblen and later by Schütte.

The helpful functions on level 2 generated by the Veblen-Schütte methods by no means exhaust $\mathbb{H}^{(2)}$. Those methods use only $[\,0\,]$ and $[\,1\,]$. We now have $[\,2\,], [\,3\,]\ldots$ and these can be combined to produce many more members of $\mathbb{H}^{(2)}$, and each of these can be use to generate a Veblen-like hierarchy.

The next few ordinals in our sequence are

$$
\begin{aligned}
\Delta[5] &= [\,3\,][\,2\,][\,1\,][\,0\,]\boldsymbol{Next}\,\omega = \text{least } \nu \text{ with } \nu = [\,2\,]^{\nu}[\,1\,][\,0\,]\boldsymbol{Next}\,\omega\\
\Delta[6] &= [\,4\,][\,3\,][\,2\,][\,1\,][\,0\,]\boldsymbol{Next}\,\omega = \text{least } \nu \text{ with } \nu = [\,3\,]^{\nu}[\,2\,][\,1\,][\,0\,]\boldsymbol{Next}\,\omega\\
\Delta[7] &= [\,5\,][\,4\,][\,3\,][\,2\,][\,1\,][\,0\,]\boldsymbol{Next}\,\omega = \text{least } \nu \text{ with } \nu = [\,4\,]^{\nu}[\,3\,][\,2\,][\,1\,][\,0\,]\boldsymbol{Next}\,\omega
\end{aligned}
$$

and so on. As far as I know these have not appeared explicitly in any method of generating ordinals.

The modern method of generating ordinals seems to be quite different. It uses an appropriate collapsing function

$$\psi : [0, \Omega^+) \longrightarrow [0, \Omega)$$

which enumerates the critical ordinals. Of course, such a function must be constant for long stretches. The precise details of ψ are not needed here. Using an iteration of the exponentiation function to base Ω, for each $l < \omega$ let

$$\nabla[l] = \psi\big((\Omega^\bullet)^l 0\big)$$

to obtain a fundamental sequence $\nabla[\cdot]$ for the Howard ordinal. For instance

$$\nabla[0] = \psi 0 \quad \nabla[1] = \psi 1 \quad \nabla[2] = \psi\Omega \quad \nabla[3] = \psi(\Omega^\Omega) \quad \nabla[4] = \psi(\Omega^{\Omega^\Omega})$$

and so on. As in [7], with a bit of effort it can be shown that $\nabla[\cdot]$ and $\Delta[\cdot]$ are essentially the same sequence, that is

$$\Delta[0] = \omega < \epsilon_0 = \nabla[0] \quad \Delta[1] = \epsilon_0 < \epsilon_1 = \nabla[1] \quad \Delta[l+2] = \nabla[l+2]$$

for each $l < \omega$. As can be seen from the analysis in [7], it is not so much the size of the ξ that determines the output $\psi\xi$, but the type structure hidden in a canonical expansion of ξ to base Ω. The method of generating ordinals 'from below' simply makes that type structure and associated gadgetry more explicit.

Let me conclude with some remarks on the work of Setzer as described in [4].

Here and in [7] I used the phrase 'from below' to describe the method of naming ordinals based on iterates of the helpful functions $[\,\imath\,]$. This is merely a convenient way of distinguishing that method from the modern method, which I described as 'from above'. Nevertheless, there is clearly a more fundamental difference between the two methods. In [4] Setzer puts more meat on the skeletal phrase 'from below'. That work is clearly an important step towards making the difference between the two methods quite precise.

As part of the analysis he uses extended Schütte brackets in which weaker version are nested to produce more powerful versions. It can be shown that the standard Schütte brackets are essentially those helpful functions that can be built in a certain way from $[\,0\,]$ and $[\,1\,]$. Some of the calculations in [4] seem to suggest that the extended Schütte brackets can be built in a similar way from $[\,2\,], [\,3\,], [\,4\,], \ldots$ and so on. It would be interesting to see the details worked out.

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